Circular 3-linkages in a hexagon.

Problem with a solution proposed by Arkady Alt , **San Jose** , **California**, **USA**.

Let a_i is number of rings on the rod at the vertices $i = 1,2,\ldots,6$ of a regular hexagon. Find the greatest value of 3 –rings chains constructed from rings taken by one from any 3 neighboring rods.

Solution.

Let $x_1,x_2,\ldots,x_5,x_6$ is number of 3 –chains, constructed from rings taken by one from rods at pair of vertices $(1,2),(2,3),\ldots,(5,6),(6,1)$ respectively.

Then we should find $C_3(a_1,a_2,\ldots,a_6)$ which denote the greatest value of sum $x_1+x_2+\ldots+x_6$ if nonnegative integer numbers $x_1,x_2,\ldots,x_5,x_6$ satisfy to inequalities

(**1**) $$\begin{cases} x_6+x_1+x_2 \le a_1 \\ x_{i-1}+x_i+x_{i+1} \le a_i, i=2,3,4. \\ x_5+x_6+x_1 \le a_6 \end{cases}.$$

Let $s := x_1+x_2+\ldots+x_6, t_1 := x_6+x_1+x_2, t_i := x_{i-1}+x_i+x_{i+1}, t_6 := x_5+x_6+x_1$ then

(**2**) $$\begin{cases} t_1+t_4 = t_2+t_5 = t_3+t_6 \\ t_i \le a_i, i=1,2,..,6. \end{cases}$$

and $t_1+t_4 = t_2+t_5 = t_3+t_6 = s.$

Since $x_2 = t_1-x_6-x_1, x_5 = t_6-x_6-x_1, x_3 = t_2-x_1-x_2 = t_2-x_1-t_1+x_6+x_1 = t_2-t_1+x_6,\ x_4 = t_5-x_5-x_6 = t_5-t_6+x_6+x_1-x_6 = t_5-t_6+x_1$ and $x_2,x_3,x_4,x_5 \ge 0$ we have

$$\begin{cases} x_6+x_1 \le \min\{t_1,t_6\} \\ t_6-t_5 \le x_1 \\ t_1-t_2 \le x_6 \end{cases} \Leftrightarrow \begin{cases} t_1-t_2 \le x_6 \le \min\{t_1,t_6\}-x_1 \\ t_6-t_5 \le x_1 \\ t_1-t_2 \le \min\{t_1,t_6\}-x_1 \end{cases} \Leftrightarrow$$

$$\begin{cases} t_6-t_5 \le x_1 \le \min\{t_1,t_6\}-t_1+t_2 \\ t_6-t_5 \le \min\{t_1,t_6\}-t_1+t_2 \\ t_1-t_2 \le x_6 \le \min\{t_1,t_6\}-x_1 \end{cases} \Leftrightarrow \begin{cases} t_6+t_1-t_2-t_5 \le \min\{t_1,t_6\} \\ t_6-t_5 \le x_1 \le \min\{t_1,t_6\}-t_1+t_2 \\ t_1-t_2 \le x_6 \le \min\{t_1,t_6\}-x_1 \end{cases}$$

Since $t_2+t_5 = t_1+t_4 = t_3+t_6$ yields $\max\{t_1,t_6\} \le t_2+t_5 \Leftrightarrow t_6+t_1-\min\{t_1,t_6\} \le t_2+t_5$ then

(**3**) $$\begin{cases} t_6-t_5 \le x_1 \le \min\{t_1,t_6\}-t_1+t_2 \\ t_1-t_2 \le x_6 \le \min\{t_1,t_6\}-x_1 \end{cases} \Leftrightarrow \begin{cases} x_6+x_1 \le \min\{t_1,t_6\} \\ t_6-t_5 \le x_1 \\ t_1-t_2 \le x_6 \end{cases}$$

for any nonnegative integer numbers t_1,t_2,t_3,t_4,t_5,t_6 satisfying (**2**).

Therefore, integer numbers x_1,x_2,x_3,x_4,x_5,x_6 where x_1,x_6 satisfy (**3**) and $x_2 = t_1-x_6-x_1$, $x_3 = t_2-t_1+x_6, x_4 = t_5-t_6+x_1,\ x_5 = t_6-x_6-x_1$ are nonnegative solutions of the system

(3) $$\begin{cases} x_6 + x_1 + x_2 = t_1 \\ x_{i-1} + x_i + x_{i+1} = t_i, i = 2,3,4 \\ x_5 + x_6 + x_1 = t_6 \end{cases}.$$

Thus, original problem equivalent to finding greatest value of parameter s for which solvable the system

(4) $$\begin{cases} t_1 + t_4 = t_2 + t_5 = t_3 + t_6 = s \\ 0 \le t_i \le a_i, i = 1,2,..,6. \end{cases}.$$

Since $t_{i+3} = s - t_i, i = 1,2,3$ then $s - t_i \le a_{i+3} \Leftrightarrow s - a_{i+3} \le t_i, i = 1,2,3$ and

(4) $$\Leftrightarrow \begin{cases} \max\{0, s - a_4\} \le t_1 \le a_1 \\ \max\{0, s - a_5\} \le t_2 \le a_2 \\ \max\{0, s - a_6\} \le t_3 \le a_3 \\ t_{i+3} = s - t_i, i = 1,2,3 \end{cases} \Leftrightarrow \begin{cases} \max\{0, s - a_{i+3}\} \le t_i \le a_i, i = 1,2,3 \\ t_{i+3} = s - t_i, i = 1,2,3 \\ s - a_{i+3} \le a_i, i = 1,2,3 \end{cases}.$$

Hence, system (**4**) is solvable iff

$s \le a_i + a_{i+3}, i = 1,2,3 \Leftrightarrow s \le \min\{a_1 + a_4, a_2 + a_5, a_3 + a_6\}$

and, therefore, $C_3(a_1, a_2, \ldots, a_6) = \max s = \min\{a_1 + a_4, a_2 + a_5, a_3 + a_6\}$.